Conservation of momentum in electrodynamics and *momentum density* of electromagnetic fields:

$\vec{F}$ is the force exerted by the electromagnetic fields on the charge distribution in volume τ. It will be equal to the rate of change of mechanical momentum, $\vec{P}_{mech}$ (say), of the charge distribution. Therefore eqn. (13B) can be written as:

$$\vec{F} = \frac{d\vec{P}_{mech}}{dt} = \oint\!\!\!\!\oint_S \overleftrightarrow{T} \cdot d\vec{a} - \mu_0\epsilon_0 \frac{d}{dt}\int_\tau \vec{S}d\tau$$

Or rearranging:

$$\frac{d}{dt}\left(\vec{P}_{mech} + \int_\tau \mu_0\epsilon_0\vec{S}d\tau\right) = \oint\!\!\!\!\oint_S \overleftrightarrow{T} \cdot d\vec{a} \ldots\ldots\ldots (15)$$

Clearly, $\int_\tau \mu_0\epsilon_0\vec{S}d\tau$ has the dimension of momentum. We can interpret it as the momentum of the electromagnetic field, stored in the volume τ. Then the left hand side of eqn. (15) is the rate of increase of total momentum in the volume τ. Consequently, as indicated by the principle of conservation of linear momentum, the right hand side is the momentum flow per second into the volume τ across its boundary surface S. Thus $-\overleftrightarrow{T}$ can be interpreted as the outward momentum flux density across S.

If we write $\vec{p}_{mech}$ for the mechanical momentum per unit volume then

$$\vec{P}_{mech} = \int_\tau \vec{p}_{mech}d\tau$$

And equation (15) reduces to:

$$\frac{d}{dt}\int_\tau (\vec{p}_{mech} + \mu_0\epsilon_0\vec{S})d\tau = \oint\!\!\!\!\oint_S \overleftrightarrow{T} \cdot d\vec{a} \ldots\ldots\ldots (16)$$

Thus $\mu_0\epsilon_0\vec{S}$ Has the dimension of momentum per unit volume. We have identified $\int_\tau \mu_0\epsilon_0\vec{S}d\tau$ as the momentum of the electromagnetic fields. Therefore ***$\mu_0\epsilon_0\vec{S}$ is the momentum density of the electromagnetic fields***. We may call it $\vec{p}_{e.m}$. If we write $\vec{p} = \vec{p}_{mech} + \vec{p}_{e.m}$ as the total momentum density then, eqn. (16) reduces to:

$$\frac{d}{dt}\int_\tau (\vec{p}_{mech} + \vec{p}_{e.m})d\tau = \frac{d}{dt}\int_\tau \vec{p}d\tau = \oint\!\!\!\!\oint_S \overleftrightarrow{T} \cdot d\vec{a} \ldots\ldots\ldots (17)$$

With the help of Gauss's divergence theorem:

$$\frac{d}{dt}\int_\tau (\vec{p}_{mech} + \vec{p}_{e.m})d\tau = \frac{d}{dt}\int_\tau \vec{p}d\tau = \int_\tau \vec{\nabla} \cdot \overleftrightarrow{T}d\tau$$

$$\Rightarrow \int_{\tau} \left(\frac{d\vec{\boldsymbol{p}}}{dt} - \vec{\boldsymbol{\nabla}} \cdot \overleftrightarrow{\boldsymbol{T}} \right) d\tau = 0$$

And since $d\tau$ is arbitrary, we can write:

$$\frac{d}{dt}(\vec{\boldsymbol{p}}_{\boldsymbol{mech}} + \vec{\boldsymbol{p}}_{\boldsymbol{e.m}}) + \vec{\boldsymbol{\nabla}} \cdot (-\overleftrightarrow{\boldsymbol{T}}) = \frac{d\vec{\boldsymbol{p}}}{dt} + \vec{\boldsymbol{\nabla}} \cdot (-\overleftrightarrow{\boldsymbol{T}}) = 0 \dots\dots\dots (18)$$

Comparing with the equation of conservation of energy in electrodynamics (Poynting theorem):

$$\frac{d}{dt}(U_{mech} + U_{e.m}) + \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{S}} = \frac{dU}{dt} + \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{S}} = 0$$

where U_{mech} & $U_{e.m}$ are the mechanical energy density and energy density of the electromagnetic field and $\vec{\boldsymbol{S}}$ is the Poynting vector, representing energy flowing normally across unit area per second, we can name equation (18) as the law of conservation of momentum in an electromagnetic field. Thus **we can term $-\overleftrightarrow{\boldsymbol{T}}$ as** the momentum flowing normally across unit area per second, i.e. **momentum flux density**. The same interpretation of $\overleftrightarrow{\boldsymbol{T}}$ is evident from equation (17), where $\oiint_S \overleftrightarrow{\boldsymbol{T}} \cdot \boldsymbol{d\vec{a}}$ represents the momentum entering the volume τ per second crossing the boundary surface of τ normally at every point. The element $-T_{ij}$ can be interpreted as the momentum flowing per unit area along direction $\boldsymbol{i}$ across an area oriented in direction $\boldsymbol{j}$.

Thus we interpreted $\overleftrightarrow{\boldsymbol{T}}$ in two different ways. From equations (13A & B) and (14), $\overleftrightarrow{\boldsymbol{T}}$ is interpreted as the stress tensor i.e. the electromagnetic stress acting on a surface. While from equations (15) and (19) $-\overleftrightarrow{\boldsymbol{T}}$ is interpreted as the momentum flux density the momentum transported by the fields per second per unit area across the surface. Similarly we also interpreted $\vec{\mathbf{S}}$ in two different ways. Poynting theorem interprets $\vec{\mathbf{S}}$ as energy flux density i.e. energy transported across unit area per second by the electromagnetic fields while equation (16) interprets $\boldsymbol{\mu_0 \epsilon_0} \vec{\mathbf{S}}$ as the momentum stored per unit volume in the fields.

Angular Momentum:

We have already seen that electromagnetic fields are not only the mediators of the force between charges (static or moving), but they themselves carry energy (per unit volume)

$$\boldsymbol{U_{e.m}} = \frac{\mathbf{1}}{\mathbf{2}}\left(\boldsymbol{\epsilon_0 E^2} + \frac{\mathbf{1}}{\boldsymbol{\mu_0}}\boldsymbol{B^2}\right)$$

and momentum (per unit volume)

$$\vec{\boldsymbol{p}}_{\boldsymbol{e.m}} = \boldsymbol{\mu_0 \epsilon_0} \vec{\mathbf{S}} = \boldsymbol{\epsilon_0} \vec{\mathbf{E}} \times \vec{\mathbf{B}}.$$

We can also say that the fields have **angular momentum per unit volume**:

$$\vec{\boldsymbol{l}}_{\boldsymbol{e.m}} = \vec{\boldsymbol{r}} \times \vec{\boldsymbol{p}}_{\boldsymbol{e.m}} = \boldsymbol{\epsilon_0}[\vec{\boldsymbol{r}} \times (\vec{\mathbf{E}} \times \vec{\mathbf{B}})].$$